# Assessment of Classical and Bayesian approach for Estimation of Structural changes in Panel Data

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**Abstract:** The current research focuses on different general modeling approaches to determine the presence or absence of multiple change points in each row of the 2-D data congregated at numerous time points on different subjects. Two approaches, Classical and Bayesian were implemented to estimate the change points in the per capita income change of 50 US states observed from 1948 to 2013. The Classical approach was applied following Random effects model with three terms in consideration *viz.*, common term for all subjects, subject specific error term and individual error term. Estimation of change points were done by least square theory. The Bayesian approach was employed concerning three assumptions: (a) subjects follow normal distribution having a change in parameters after the change points, (b) a correlation exists among the parameters before and after the change point for each subject, (c) change points of different series follow a common distribution. In this approach, estimation was performed by Markov Chain Monte Carlo (MCMC) method. A comparison amid the two estimates was executed by determining the standard error. In conclusion, two change points were observed in many of the states, generally, in 1984 and 1988. Some of the states exhibited no evidence of structural changes implying diminished effect of Great Moderation. The Bayesian approach displayed better estimate over the Classical one.
**Keywords:** Panel data, Change point, Classical approach, Least square theory, Bayesian approach, Markov Chain Monte Carlo method.

## 1 Introduction

In various applications, the data obtained during an extensive time period has to be investigated representing that probable statistical model may alter once or several times during the period of surveillance. The alteration in statistical model signifies a change and the point of alteration occurrence is the change point. In a series of random variables $X_1, X_2, \dots\dots X_n$ if $X_1, X_2, \dots\dots X_\eta$ follows a common distribution F and $X_{\eta+1}, X_{\eta+2}, \dots\dots X_n$ have distribution G with F ≠ G, then, the index '$\eta$' is called the change point. The change-point problem was first introduced in the quality control context by Page [1]. The application of change point problems is found in statistical quality control theory, reliability, stochastic process, testing and estimation of change in the patterns of a regression models in statistics as well as in subjects like genetics, medicine, finance, stock market data and various others fields. Statistical exploration of change-point problems depend on the type of data to be examined. Time series data is usually of two types: one-dimensional and multi-dimensional or panel data. A 2-D data quite often, may be a panel data gathered at several time points on numeral subjects. Introduction of a shared involvement for all the themes (subjects) may lead to single or multiple change points occasionally called structural changes, in each row of panel data.

Some prominent instances are evidence of structural breaks in volatility due to Great Moderation in each of the 50 US states, change in blood pressure of patients in data recorded weekly before and after application of a certain drug, monthly number of traffic deaths on rural interstate roads for all US states from the time period of April, 1985 to April,1989 including the year in which the 55 miles per hour speed limit was lifted, etc.

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Several literature have been reported on change point analysis of panel data. A common break model with breaks occurring at the same time point for all the subjects was developed by Joseph [2]. The assumption was that the pre-break data was sampled from a common single distribution and post break data from a common distribution. Joseph and Berger [3] have taken into account the fact that time of break may not be same for all the subjects and developed a model with the assumption that change points are not identical; rather they are samples from a common distribution function with parameter**.** Model with common break for all subjects and the break point being estimated by the theory of least squares was suggested by Bai and Perron [4] along with the consistency and asymptotic distribution of the estimate. The method was also extended for partial structural change model where not all the parameters but only a few are subject to shifts due to structural change. An efficient dynamic programming algorithm was proposed by Bai and Perron [5] to obtain global minimizers of the sum of squared residuals for both pure and partial structural change model. A simulation analysis was carried out by Bai and Perron [6] to access the adequacy of the previously proposed methods and compare between their relative merits, size and power of tests for structural change. Bai [7] studied the problem of common breaks in heterogeneous means and variances in panel data i.e., magnitude of change for the series depend on index, but assumed that breaks for all the subjects occur at the same time.

However a more realistic model is that the break points in the panel data occur at different time points for each of the subjects and common break points would be a special case of the more realistic model. More practical assumptions are (i) the change points of different subjects do not occur at the same time and (ii) the magnitude of break depends on the subjects i.e., the amount of change may not be same for all the subjects. A model for US data on quarterly personal income in million dollars of states from 1$^{st}$ quarter of the year 1948 to the third quarter of the year 2012 during Great Moderation has been proposed by taking into account the above mentioned assumptions. But this model has an intrinsic assumption that observations are independent in each row of the data i.e., the observations taken over regular time interval for the same subject are independent of each other. This assumption is not at all true, as the internal characteristic of the subject is the common internal aspect that is existent among all the observations recorded over time on a particular subject. The elimination of this common internal factor will end in independent observations along each row of the panel data. Analyzing these independent observations acquired after removal of the internal shared factor for each subject to estimate their change point will result in an improved and more precise estimation of change point for the panel data.

In the present research article, a methodology to eliminate internal communal factor for each subject have been proposed and the change point analysis of the resulting independent data have been accomplished under the assumption of heterogeneity in the timings of structural changes. Assessment of change point is done in both Classical and Bayesian approach. The Classical method was applied following one way Random effects model with three terms in consideration *viz.*, common term for all subjects signifying the common effect of the whole panel data, subject specific error term symbolizing the common internal effect of each subject and individual error term. Change points were calculated using the theory of least squares. The Bayesian tactic was hired concerning three assumptions: (a) subjects follow normal distribution having a change in parameters after the change points, (b) a correlation exists among the parameters before and after the change point for each subject which represents the internal common effect of each subject before and after the change point, (c) change points of different series follow a common distribution signifying common effect of the whole panel data. In this method, assessment was executed by Markov Chain Monte Carlo (MCMC) method. Comparison amid the two methods is also exhibited.

The rest of the research article is organised as follows: Section 2 discusses the general structure of panel data in detail particularly that of the US state level data from 1948 first quarter to 2012 third quarter on which the model of interest is based. Section

3 defines our model and Section 4 describes the methodology to eradicate internal communal factor for each subject of the panel data, whereas Section 5 and Section 6 demonstrates the analysis by Classical methodology and Bayesian methodology respectively. Section 7 explains the results and the comparisons and section 8 concludes the paper.

# 2 Panel data

The structure of the panel data has the form of an $\boldsymbol{N}$ **x** $\boldsymbol{T}$ matrix $\boldsymbol{X}$ as follows. Each sequence $X_{i1}, X_{i2}, \ldots\ldots, X_{iT}$ represents observations over time from the $i^{th}$ subject, $i = 1, \ldots, N$. A change is said to have occurred at $\tau_i$ in sequence or row $i$ , $i = 1, \ldots., N$, and $1 \leq \tau_i \leq N - 1$ , if $X_{i1}, X_{i2}, \ldots., X_{i\tau_i}$ are identically distributed with common distribution function $F_{i1}$ and $X_{i\tau_i+1}, X_{i\tau_i+2}, \ldots., X_{iN}$ are identically distributed with common distribution function $F_{i2}$ with $F_{i1} \neq F_{i2}$ . If $\tau_i = T$, then there is no change in row '$i$'. Thus $\tau_1, \tau_2, \ldots. \tau_N$ gives the change points which are generally unknown. The distributions of the points of change, '$\tau_i$', and the unknown parameters of the distributions $F_{ik}$, $i = 1, \ldots\ldots, N; k = 1,2$ are to be estimated from the data matrix (1).

$$\boldsymbol{X} = \begin{pmatrix} X_{11}\ X_{12} & \ldots. & X_{1\tau_1}\ X_{1\tau_1+1} & \ldots. & X_{IT} \\ X_{21}\ X_{22} & \ldots. & X_{2\tau_2}\ X_{2\tau_2+1} & \ldots. & X_{2T} \\ \ldots\ \ldots & \ldots. & \ldots\ \ldots & \ldots. & \ldots \\ \ldots\ \ldots & \ldots. & \ldots\ \ldots & \ldots. & \ldots \\ \ldots\ \ldots & \ldots. & \ldots\ \ldots & \ldots. & \ldots \\ \ldots\ \ldots & \ldots. & \ldots\ \ldots & \ldots. & \ldots \\ \ldots\ \ldots & \ldots. & \ldots\ \ldots & \ldots. & \ldots \\ X_{N1}\ X_{N2} & \ldots. & X_{N\tau_N}\ X_{N\tau_N+1} & \ldots. & X_{NT} \end{pmatrix} \qquad \ldots\ldots. (1)$$

**Instance:** During 1980s, the major United States macroeconomic time series of output growth and inflation have experienced a considerable decrease in volatility. This is generally talked about to as Great Moderation. The origin behind this reduction may be that better monetary policy steadied the economy or may be upgraded inventory control facilitated the stabilization. The data is published by the Bureau of Economic Analysis, USA. A structure of the data on 50 US state level quarterly personal incomes from 1st quarter of the year 1948 to 3rd quarter of the year 2012 is given in Table 1.

**Table 1. Structure of the data on US state level quarterly total personal income in US million dollars from 1st quarter of 1948 to 3rd quarter of 2012**

| | **1948 Q1** | **1948 Q2** | **1948 Q3** | **1948 Q4** | **1949 Q1** | **……….** | **2012 Q2** | **2012 Q3** |
|---|---|---|---|---|---|---|---|---|
| **Alabama** | 2459 | 2582 | 2638 | 2681 | 2551 | ……… | 171631 | 172835 |
| **Alaska** | 297 | 311 | 333 | 356 | 388 | ……… | 33918 | 34050 |
| **Arizona** | 862 | 892 | 945 | 922 | 918 | ……… | 235331 | 236833 |
| **Colorado** | 1752 | 1807 | 1858 | 1852 | 1804 | ……… | 233400 | 234776 |
| **…..** | ……. | ……… | ……… | ……… | ……… | ……… | ……… | ………. |
| **….** | …….. | ……… | ……… | ………. | ……… | ……… | ……… | ………. |
| **West Virginia** | 2033 | 1969 | 2200 | 2188 | 2092 | ……… | 64126 | 64271 |

Observing Table 1, it can be straightforwardly understood that each cell value signifies quarterly income of each state of US, each row of the table represents each state of US and each of the time interval is denoted by each column of the panel data matrix.

One way to track down the source of the Great Moderation is to analyze the data at state level. States are subjected to conjoint nation-wide and global shocks. A recent work by Owyang *et al.* [8] described the state-level facts of Great Moderation in growth rate of unemployment. However, the researchers conducted the structural change analysis by univariate method i.e., they estimated the change point series by series without taking into account the similarity among different states. Liao [9] in his unpublished manuscript fitted a multivariate model for change point detection at the state level for data on per capita income of the states. But the assumption of normal distribution for income was not quite reasonable.

In the present research article, we at first, modelled the data on 50 US state level quarterly total personal incomes supposing the data to follow Pareto distribution. The reason behind using Pareto distribution is that it is generally taken as the typical income distribution. We assumed that per capita incomes of each state follow Pareto distribution and there is change in the parameter values of the distribution after the change point which requires to be estimated. Later on, we eliminated the common influence of a state from observations over time in each row of the panel data matrix and remodelled the data and estimated the data using Classical as well as Bayesian approach. The following Sections describe the detailed accomplishment of the work done.

## 3 The model

Considering the data on per capita quarterly income of 50 US states from 1st quarter of the year 1948 to 3rd quarter of the year 2012 we have, $N = 50$ and $T = 259$.

The growth in log of income is computed for state $i$, $i = 1,2, \ldots 50$ and time point $t$, $t = 1,2, \ldots .259$ as,

$$X_{it} = log(personal\ income_{it}) - log(personal\ income_{it-1})$$

We made the following assumptions for our model:

i. Series *i* experiences a single structural break at time $\tau_i$ , $i=1,2.. N$
ii. $\tau_i \neq \tau_j$, for some $i \neq j$ , that is, structural break does not occur at the same time for all the *N* series.
iii. Magnitude of change occurred may vary for each series.
iv. For series $i$,

$$X_{it} \sim Pareto(\alpha_{i1}) \quad for \ \ t \leq \tau_i$$
$$X_{it} \sim Pareto(\alpha_{i2}) \quad for \ \ \tau_i + 1 < t \leq T - 1 \qquad \ldots\ldots. (2)$$

Therefore, $\theta = \{\alpha_{ij}\}, = 1, 2\ ;\ = 1,2 \ldots..N\ ;\ t = 1,2 \ldots.T$ are the unknown model parameters along with $\tau = (\tau_1, \tau_2, \ldots..\tau_N)$ , vector of change points assumed to follow normal distribution i.e.,

$$\tau_i \sim N\ (\mu_i, \sigma_i^2)\ ;\ i = 1,2, \ldots.N$$

The model parameters $\{\vec{\theta}\ , \vec{\mu}, \overrightarrow{\sigma^2}\}$ need to be estimated.

Initial values of $\tau_i$*'s* are chosen from growth rate graph of the states. As for example, for the state Colorado, the graph in Fig. 1 is as follows:

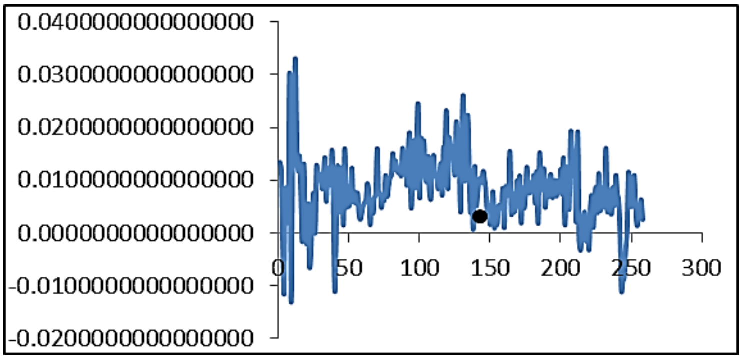


**Fig. 1. Income growth in terms of difference in logarithm for the state Colorado**

Hence, from the above graph, the initial estimate of $\tau_i$ is taken to be 145 *i.e.*, 3rd quarter of 1980. The model parameters were estimated by Gibbs Sampling and Markov Chain Monte Carlo technique. The steps of estimation are as follows:

**Step 1:** Given initial values $\left(\alpha_{ij}^{(0)}, \tau_i^{(0)}, \mu_i^{(0)}, \sigma_i^{(0)}\right)$, $i = 1,2, \dots .50;\ j = 1,2.$

**Step 2:** Update parameters $\alpha_{ij}$ , $j = 1,2.$

Given initial values $\left(\alpha_{ij}^{(0)}, \tau_i^{(0)}, \mu_i^{(0)}, \sigma_i^{(0)}\right)$, $i = 1,2, \dots .50;\ j = 1,2$ and data matrix $\boldsymbol{X}$, the posterior conditional distribution of $\alpha_{i1}^{(1)}$ is,

$$\alpha_{i1}^{(1)} \mid \tau_i^{(0)}, \mu_i^{(0)}, \sigma_i^{(0)} \sim Gamma\ (\alpha_{ij}^{(0)} + \tau_i^{(0)}, \qquad \beta_0 + \sum_{i=1}^{\tau_i^{(0)}} \frac{X_i}{X_m})$$

and that of $\alpha_{i2}^{(1)}$ is,

$$\alpha_{i2}^{(1)} \mid \tau_i^{(0)}, \mu_i^{(0)}, \sigma_i^{(0)} \sim Gamma\ (\alpha_{ij}^{(0)} + M - \tau_i^{(0)}, \qquad \beta_0 + \sum_{i=\tau_i^{(0)}}^{M} \frac{X_i}{X_m})$$

**Step 3:** Update parameters $\tau_i$ , $i = 1,2, \dots M$

$\tau \sim f(\tau \mid \alpha, \mu, \sigma, X) \propto f\ (X \mid \alpha, \tau). f(\ \tau \mid \mu, \sigma\ )$

Draw $\tau_i^{(1)}$ given $\alpha_{ij}^{(1)}, \mu_i^{(0)}, \sigma_i^{(0)}$.

**Step 4:** Update parameters $\mu_i$ and $\sigma_i$ , $i = 1,2, \dots .50$

Draw $(\ \mu_i^{(1)}, \sigma_i^{(1)}\ )$ given $(\alpha_{ij}^{(1)}, \tau_i^{(1)}, X\ )$.

$$(\mu_i^{(1)}, \sigma_i^{(1)} \mid \tau_i^{(1)}\ ) \sim Normal - \ Inverse\ Gamma\ (\frac{(\lambda_0\mu_0\ + N\tau_{mean})}{\lambda_0} + N, \lambda_0\ + N, \gamma_0 + \frac{N}{2}\ , \delta_0 + \frac{1}{2}\sum_{i=1}^{M}(\tau_i - \tau_{mean})^2 + \frac{N\lambda_0(\tau_{mean} - \mu_0)^2}{2(\lambda_0 + N)})$$

But the dependence between $\alpha_{i1}$ and $\alpha_{i2}$ in (2) for each $i = 1, 2 \dots .. N$ is not taken into account in this model. Hence we go for the next model which is a more generalized one.

A generalised panel data model is given by,

$$y_{it} = \alpha + X_{it}'\beta + u_{it} \qquad i = 1,2, \dots . N, \quad t = 1,2, \dots .. T \qquad \dots \dots (3)$$

where, $i$ denote subjects present in each row of the panel data e.g., households, individuals, countries, etc.; $t$ denotes the time intervals at which observations are realised; $\alpha$ denotes the scalar that symbolises the common effect of panel data which is present in all observations, $\beta_{k\times 1}$ denotes the vector of parameters are may have altered after the change point, $X_{it}$ denotes the $i^{th}$ observation on $k$ explanatory variables (covariates) at time $t$ and $u_{it}$ denotes the error term corresponding to $i^{th}$ observation on $k$ explanatory variables at time $t$.

Now as observations for each subject over time are interdependent, so the error term $u_{it}$ can be splitted as,

$$u_{it} = \mu_i + v_{it} \quad i = 1,2, \ldots . N, \quad t = 1,2, \ldots . . T \qquad \ldots\ldots.. (4)$$

where, $\mu_i$ denotes unobservable individual specific effect for each of the subjects that leads to correlation among observations in each row and $v_{it}$ denotes the purely random disturbance term.

Our problem is to eliminate the unknown $\mu_i$ term which will result in independent observations across each row of the panel data.

## 4 Elimination of Communal effect

As $\mu_i$'s are unknown and random quantities, it is reasonable to assume that $\mu_i's$ are independent and identically distributed with $\mu_i \sim (0, \sigma_\mu^2)$ and also the random disturbance term $v_{it}$ 's are independent and identically distributed with $v_{it} \sim Normal(0, \sigma_v^2)$. Another practical assumption is that $\mu_i$'s are independent of $v_{it}$'s and the covariate matrix $X_{it}$ is independent of $\mu_i$ and $v_{it}$. Hence the model is a **one way random effects model.** Inference pertains to large population from which the sample is drawn. Writing equation (3) as,

$$y = \alpha l_{NT} + X\beta + u = Z\delta + u \qquad \ldots\ldots\ldots\ldots (5)$$

where, $y$ is the $NT \times 1$ response vector, $X$ is $NT \times k$ covariate matrix, $Z_{NT\times k}$ is the augmented matrix of the form $Z = [l_{NT}\ X]$, $\delta'$ denotes the set of unknown parameters $(\alpha'\beta')$ and $l_{NT}$ is vector of ones of dimension $NT$.

Therefore $u = (u_{11}, u_{12}, \ldots . u_{1T}, u_{21}, u_{22}, \ldots . u_{2T}, \ldots \ldots \ldots u_{N1}, u_{N2}, \ldots . u_{NT})$ in (4) can be written in the form ,

$$u = Z_\mu \mu + v \ \ldots\ldots\ldots\ldots\ldots (6)$$

where, $v = (v_{11}, v_{12}, \ldots . v_{1T}, v_{21}, v_{22}, \ldots . v_{2T}, \ldots \ldots \ldots v_{N1}, v_{N2}, \ldots . v_{NT})$ and $\mu' = (\mu_1, \mu_2, \ldots . , \mu_N)$ and $Z_\mu = I_N \otimes l_T$, $I_N$ being the identity matrix of order $N \times N$ and $l_T$ is a vector of ones of dimension $T$.

Thus we have the final equation as,

$$y = Z\delta + Z_\mu \mu + v \ \ldots\ldots\ldots\ldots (7)$$

Now $Z_\mu{}' Z_\mu = I_N \otimes J_T$ where $J_T$ is a matrix of ones of dimension $T \times T$.

Denoting $\Omega$ as variance-covariance matrix of $u$ we have,

$$\Omega = E(uu') = Z_\mu E(\mu\mu') + \sigma_v^2 I_{NT}$$

With

$$Var(u_{it}) = \sigma_\mu^2 + \sigma_v^2 \ for\ all\ i, t$$
$$Cov(u_{it}, u_{js}) = \sigma_\mu^2 + \sigma_v^2 \ for\ i = t, j = s$$
$$= \sigma_\mu^2 \ for\ i = t, j \neq s$$
$$= 0 \ otherwise$$

In order to obtain the generalized least squares estimator of the regression coefficients, $\Omega$ is required. But $\Omega$ is a huge matrix of dimension $NT \times NT$.

A suitable technique is replacing $J_T$ by $T\bar{J}_T$, $I_T$ by $E_T + \bar{J}_T$. Hence $E_T =, I_T - \bar{J}_T$.

So, $\Omega = \mathrm{T}\sigma_\mu^2(I_N \otimes \bar{J}_T) + \sigma_v^2(I_N \otimes E_T) + \sigma_v^2(I_N \otimes \bar{J}_T)$

Collecting terms with same matrices, $\Omega = \left(\mathrm{T}\sigma_\mu^2 + \sigma_v^2\right)(I_N \otimes \bar{J}_T) + \sigma_v^2(I_N \otimes E_T)$

$$= \sigma_1^2 P + \sigma_v^2 Q$$

where, $\sigma_1^2 = T\sigma_\mu^2 + \sigma_v^2 = 1^{st}\ unit\ characteristic\ root\ of\ \Omega$ of multiplicity $N$ and $\sigma_v^2 = 2^{nd}\ unit\ characteristic\ root\ of\ \Omega$ of multiplicity $N(T-1)$

Now, $\Omega^{-1} = \frac{1}{\sigma_1^2}P + \frac{1}{\sigma_v^2}Q$ implying $\Omega^{-\frac{1}{2}} = \frac{1}{\sigma_1}P + \frac{1}{\sigma_v}Q$ implying $\sigma_v\Omega^{-\frac{1}{2}} = \frac{\sigma_v}{\sigma_1}P + Q$

Taking the transformation, $y^* = \sigma_v\Omega^{-\frac{1}{2}}y$, $Z^* = \sigma_v\Omega^{-\frac{1}{2}}Z$, following equation is obtained,

$$\boldsymbol{y_{it}^* = Z^*\delta_{it} + v_{it}^*\ for\ i = 1, 2, \dots .N\ ; t = 1, 2, \dots .T} \qquad \dots\dots\dots(8)$$

Hence the observations within each row are independent of each other as after the transformation the term $\mu_i$ is removed from the model. But, there may be dependency among the rows (as there may be some common factor affecting all the subjects in the panel) e.g., how different patients suffering from the same disease respond to a new treatment; when the same treatment being given to all the patients. This dependency is taken care off during estimation of change points.

## 5 Classical method of estimation

In Classical strategy, change points were computed applying the theory of least squares. The dependency among the subjects was taken care off by the following two ways:

i. Assuming common break for all the rows of the panel
ii. Assuming each row has break points at different times.

Under the assumption of common break and considering two breaks present in the data for each row, we have the model for each of the $N$ subject, $i = 1,2, \dots ..N$ as,

$$y_{it}^* = Z^*\delta_{it_1} + v_{it}^* \quad t = 1,2, \dots ..k_1$$
$$y_{it}^* = Z^*\delta_{it_2} + v_{it}^* \quad t = k_1+1, k_1+2, \dots ..k_2$$
$$y_{it}^* = Z^*\delta_{it_3} + v_{it}^* \quad t = k_2+1, k_2+2, \dots ..T$$

Then the sum of squared residuals for the equation in the $i^{th}$ row is given by,

$$S_{iT}(k_1, k_2) = \sum_{t=1}^{k_1}\left(y_{it}^* - Z^*\delta_{it_1}\right)^2 + \sum_{t=k_1+1}^{k_2}\left(y_{it}^* - Z^*\delta_{it_2}\right)^2 + \sum_{t=k_2+1}^{T}\left(y_{it}^* - Z^*\delta_{it_3}\right)^2$$

where, $k_1 = 2,3, \dots\dots T-1; k_2 = k_1+1, \dots\dots .T-1\ ; i = 1,2, \dots\dots .N$

Total sum of squared residuals for all the $N$ equations is as follows,

$$SSR(k_1, k_2) = \sum_{i=1}^{N} S_{iT}(k_1, k_2)$$

Thus the least square estimator for the break points in the common break model for the panel data is obtained as,

$$\left(\widehat{k_1}, \widehat{k_2}\right) = argmin_{2 \le k_1 \le T-1; k_1 < k_2 \le T-1}\ SSR(k_1, k_2) \qquad \dots\dots\dots(9)$$

Under the assumption of break points at different times and considering two breaks present in the data for each row, we have the model for each of the $N$ subject, $i = 1, 2, \dots ..N$ as,

$$y_{it}^* = Z^*\delta_{it_1} + v_{it}^* \quad t = 1,2,\ldots\ldots k_{i_1}$$
$$y_{it}^* = Z^*\delta_{it_2} + v_{it}^* \quad t = k_{i_1}+1, k_{i_1}+2,\ldots\ldots k_{i_2}$$
$$y_{it}^* = Z^*\delta_{it_3} + v_{it}^* \quad t = k_{i_2}+1, k_{i_2}+2,\ldots\ldots T$$

Then the sum of squared residuals for the equation in the $i^{th}$ row is given by,

$$S_{iT}(k_{i_1}, k_{i_2}) = \sum_{t=1}^{k_{i_1}} (y_{it}^* - Z^*\delta_{it_1})^2 + \sum_{t=k_{i_1}+1}^{k_{i_2}} (y_{it}^* - Z^*\delta_{it_2})^2 + \sum_{t=k_{i_2}+1}^{T} (y_{it}^* - Z^*\delta_{it_3})^2$$

where, $k_{i_1} = 2,3,\ldots\ldots T-1; k_{i_2} = k_1+1,\ldots\ldots.T-1\,; i = 1,2,\ldots\ldots.N$

Thus the least square estimator for the break points in the $i^{th}$ row for the panel data model is acquired as,

$$(\widehat{k_{i_1}}, \widehat{k_{i_2}}) = argmin_{2\le k_{i_1}\le T-1; k_1<k_{i_2}\le T-1} S_{iT}(k_{i_1}, k_{i_2}) \qquad \ldots\ldots\ldots\ldots(10)$$

## 6 Bayesian method of estimation

In Bayesian strategy, change points were figured applying the Markov Chain Monte Carlo (MCMC) technique. The dependency among the subjects was taken care off by the following two ways:

i. Assuming common break for all the rows of the panel
ii. Assuming each row has break points at different times.

Under the assumption of common break and considering two breaks present in the data for each row, we have the model for each of the $N$ subject, $i = 1,2,\ldots\ldots N$ as,

$$y_{it}^* \sim iid\ Normal(Z^*\delta_{it_1}, \sigma_v^2) for\ t \le k_1$$
$$y_{it}^* \sim iid\ Normal(Z^*\delta_{it_2}, \sigma_v^2) for\ k_1+1 < t \le k_2$$
$$y_{it}^* \sim iid\ Normal(Z^*\delta_{it_3}, \sigma_v^2) for\ k_2+1 < t < T$$

Likelihood for the series $i$ is given by,

$$f(y_i^*; \delta_{it_1}, \delta_{it_2}, \delta_{it_3}, \sigma_v^2, k_1, k_2) \propto \prod_{t=1}^{k_1} e^{-\frac{1}{2\pi\sigma_v}(y_{it}^* - Z^*\delta_{it_1})^2} \prod_{t=k_1+1}^{k_2} e^{-\frac{1}{2\pi\sigma_v}(y_{it}^* - Z^*\delta_{it_2})^2} \prod_{t=k_2+1}^{T} e^{-\frac{1}{2\pi\sigma_v}(y_{it}^* - Z^*\delta_{it_3})^2}$$

Hence, likelihood for the panel data is given by,

$$f(y^*; \delta_{it_1}, \delta_{it_2}, \delta_{it_3}, \sigma_v^2, k_1, k_2) \propto \prod_{i=1}^{N} \left\{ \prod_{t=1}^{k_1} e^{-\frac{1}{2\pi\sigma_v}(y_{it}^* - Z^*\delta_{it_1})^2} \prod_{t=k_1+1}^{k_2} e^{-\frac{1}{2\pi\sigma_v}(y_{it}^* - Z^*\delta_{it_2})^2} \prod_{t=k_2+1}^{T} e^{-\frac{1}{2\pi\sigma_v}(y_{it}^* - Z^*\delta_{it_3})^2} \right\} \qquad \ldots\ldots\ldots (11)$$

From this likelihood, the common change points $k_1$ and $k_2$ and other unknown parameters were estimated by the MCMC technique.

Under the assumption of break points at different times and considering two breaks present in the data for each row, we have the model for each of the $N$ subject, $i = 1,2,\ldots\ldots N$ as,

$$y_{it}^* \sim iid\ Normal(Z^*\delta_{it_1}, \sigma_v^2) for\ t \le k_{i_1}$$

$$y_{it}^{*} \sim iid\ Normal(Z^{*}\delta_{it_2}, \sigma_v^2) for\ k_{i_1} + 1 < t \le k_{i_2}$$
$$y_{it}^{*} \sim iid\ Normal(Z^{*}\delta_{it_3}, \sigma_v^2) for\ k_{i_2} + 1 < t < T$$

Likelihood for the row $i$ is given by,

$$f(y_i^{*}; \delta_{it_1}, \delta_{it_2}, \delta_{it_3}, \sigma_v^2, k_{i_1}, k_{i_2})$$
$$\propto \prod_{t=1}^{k_{i_1}} e^{-\frac{1}{2\pi\sigma_v}(y_{it}^{*}-Z^{*}\delta_{it_1})^2} \prod_{t=k_{i_1}+1}^{k_{i_2}} e^{-\frac{1}{2\pi\sigma_v}(y_{it}^{*}-Z^{*}\delta_{it_2})^2} \prod_{t=k_{i_2}+1}^{T} e^{-\frac{1}{2\pi\sigma_v}(y_{it}^{*}-Z^{*}\delta_{it_3})^2}$$
……… (12)

From the above likelihood, the common change points $k_{i_1}$ and $k_{i_2}$ and other unknown parameters for each row were estimated by the MCMC technique as before.

# 7 Results and Discussion

## 7.1 Estimation of parameters by Gibbs sampling method for the model in equation (2)

The estimated values of the change points for each of the 50 states along with the standard errors are given in Table 2.

From Table 2, it was observed that the panel data method resulted in concentrated pattern of breaks around 1984 and 1988. The standard errors of the estimates of change points for our model were significantly less than that of single series model for each state proposed by Owyang *et al.* [8]. Our model also comparatively gave better estimates than panel data model fitted by Liao [9]. The change point estimates of some states like Delaware, Minnesota and Tennessee had large standard errors. This indicated that they had no effect of Great Moderation. The states which had small standard errors like Arizona, Arkansas, Colorado, District of Columbia, Idaho, Kansas, Kentucky, Maine, Maryland, Massachusetts, Missouri, Montana, Nebraska, Nevada, New Hampshire, New York, North Carolina, North Dakota, Vermont, West Virginia etc, had very high effect in their income due to Great Moderation. Pennsylvania was an exception whose standard error of the estimated change point increased from 3.214 (single series model) to 27.768 in our model. It was 38.626 in the model by Liao [9]. This may be due to the fact that there is more than one structural break present in these states. This necessitates the extension of our model for the investigation of multiple structural breaks.

**Table 2: Estimated change points and the standard errors for each state**

| **Alabama** | **Alaska** | **Arizona** | **Arkanas** | **California** | **Colorado** | **Connecticut** |
|---|---|---|---|---|---|---|
| 1984Q3 | 1988Q2 | 1985Q1 | 1984Q1 | 1988Q4 | 1982Q3 | 1988Q4 |
| 9.8765 | 9.1293 | 2.6754 | 3.8321 | 6.4532 | 4.3215 | 5.6219 |
| **Delaware** | **District of Columbia** | **Florida** | **Georgia** | **Hawaii** | **Idaho** | **Illinois** |
| 1983Q1 | 1969Q3 | 1988Q3 | 1985Q3 | 1991Q3 | 1983Q1 | 1987Q2 |
| 40.4532 | 2.5648 | 5.4278 | 5.1762 | 8.3651 | 4.8327 | 17.6432 |
| **Indiana** | **Iowa** | **Kansas** | **Kentucky** | **Louisiana** | **Maine** | **Maryland** |
| 1984Q4 | 2000Q3 | 1984Q1 | 1984Q3 | 1986Q4 | 1989Q2 | 1989Q1 |
| 8.7634 | 25.4265 | 2.3567 | 1.9456 | 1.9274 | 2.6345 | 2.7362 |
| **Massachusetts** | **Michigan** | **Minnesota** | **Mississippi** | **Missouri** | **Montana** | **Nebraska** |
| 1989Q1 | 1983Q2 | 1994Q1 | 1957Q1 | 1984Q4 | 1983Q3 | 1984Q2 |
| 2.8654 | 8.7369 | 14.7861 | 5.9327 | 2.8387 | 2.1758 | 1.5046 |
| **Nevada** | **New Hampshire** | **New Jersey** | **New Mexico** | **New York** | **North Carolina** | **North Dakota** |
| 1981Q3 | 1989Q1 | 1989Q1 | 1984Q2 | 1987Q3 | 1984Q4 | 1989Q1 |
| 2.5476 | 3.3277 | 4.2163 | 5.1456 | 3.8605 | 1.4677 | 0.5659 |
| **Ohio** | **Oklahoma** | **Oregon** | **Pennsylvania** | **Rhode Island** | **South Carolina** | **South Dakota** |
| 1984Q4 | 1982Q3 | 1982Q1 | 1966Q4 | 1989Q1 | 1984Q3 | 1983Q3 |
| 5.9457 | 3.397 | 3.2177 | 27.768 | 3.6058 | 4.1285 | 4.9626 |
| **Tennessee** | **Texas** | **Utah** | **Vermont** | **Virginia** | **Washington** | **West Virginia** |
| 1985Q4 | 1983Q2 | 1983Q1 | 1989Q2 | 1986Q2 | 2003Q4 | 1981Q4 |
| 13.226 | 6.9788 | 5.3124 | 2.2341 | 7.6773 | 4.6378 | 1.6541 |

As an illustration, we have again exhibited the estimated change point of Colorado graphically in Fig. 2.

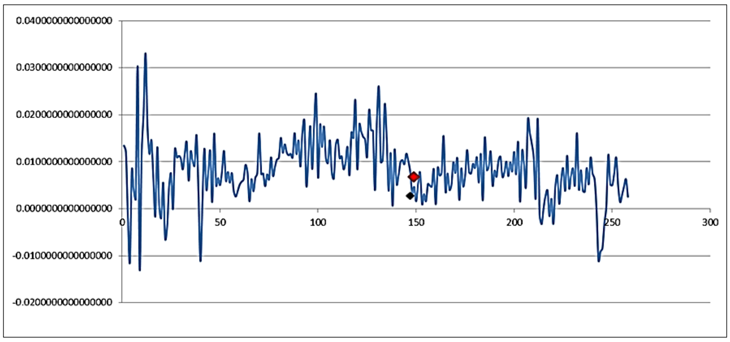


**Fig. 2. Total Income (million US dollars) graph of Colorado showing the estimated change point (red dot) and the initial value taken for Gibbs Sampling (black dot)**

### 7.2 Estimation of parameters for the model in equation (9) and (11)

Estimated common structural break for all the states by Classical and Bayesian method and the standard errors are given in the following Table 3.

**Table 3. Estimated common change points and the standard errors**

| | **Common break for all states** | |
|---|---|---|
| | **1st break** | **2nd breakbreak** |
| **Classical method** | 1984Q4<br>9.8765 | 1988Q3<br>8.7865 |
| **Bayesian method** | 1984Q4<br>6.2564 | 1988Q3<br>5.7234 |

From the Table 3, it is clear that most of the states of the panel data have two breaks, first break around the 4th quarter of 1984 and the second break around 3rd quarter of the year 1988. Bayesian estimators of the break points have lower standard errors than the Classical one, hence, Bayesian strategy gave better estimate than the classical one.

### 7.3 Estimation of parameters for the model in equation (10) and (12)

The histogram in Fig. 3 of the estimated break points for all the states exhibit two peaks, one peak around 1984 and another around 1988 and is in accordance with the conclusion drawn from the common breaks model.

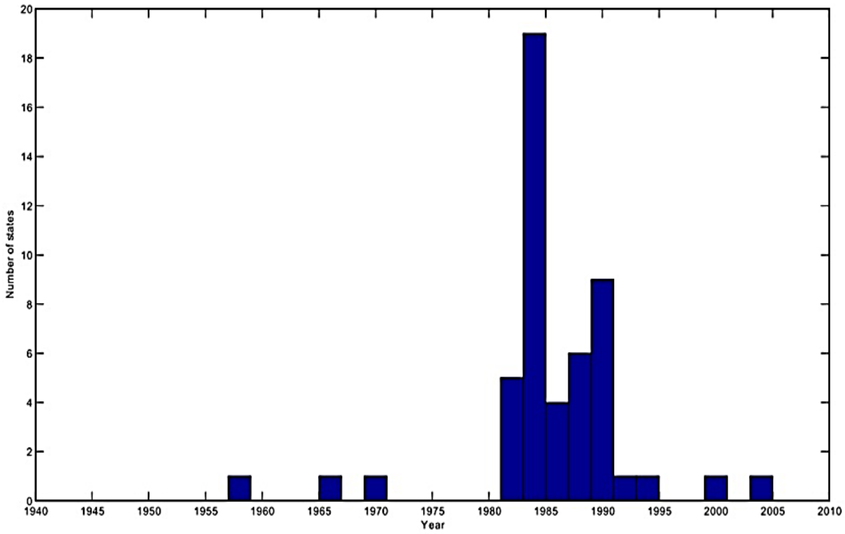


**Fig. 3. Histogram of the estimated break points for all the states**

Estimated structural breaks for each of the states by Classical and Bayesian method for each of the states is given in Table 4. The upper value for each state gave the Classical estimate while the lower value depicted the Bayesian estimate. From Table 4, it is clear that Bayesian method gives better estimates as the standard errors are low.

**Table 4. Estimated change points for different states**

| **Alabama** | | **Alaska** | | **Arizona** | | **Arkanas** | | **California** | | **Colorado** | | **Connecticut** | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1981 Q2 | 1984 Q3 | 1970 Q3 | 1988 Q2 | 1964 Q3 | 1985 Q1 | 1984 Q1 | | 1966 Q4 | 1987 Q1 | 1982 Q2 | 1997 Q2 | 1967 Q2 | 1985 Q1 |
| 1981 Q2 | 1984 Q3 | 1970 Q3 | 1988 Q2 | 1964 Q3 | 1985 Q1 | 1984 Q1 | | 1966 Q4 | 1987 Q1 | 1982 Q2 | 1997 Q2 | 1967 Q2 | 1985 Q1 |
| **Delaware** | | **District of Columbia** | | **Florida** | | **Georgia** | | **Hawaii** | | **Idaho** | | **Illinois** | |
| 1953 Q3 | 1988 Q1 | 1965 Q1 | 1988 Q3 | 1971 Q1 | 1988 Q1 | 1957 Q4 | 1983 Q1 | 1984Q3 | | 1984Q4 | | 1959 Q1 | 1984 Q3 |
| 1953 Q3 | 1988 Q1 | 1965 Q1 | 1988 Q3 | 1971 Q1 | 1988 Q1 | 1957 Q4 | 1983 Q1 | 1984Q3 | | 1984Q4 | | 1959 Q1 | 1984 Q3 |
| **Indiana** | | **Iowa** | | **Kansas** | | **Kentucky** | | **Louisiana** | | **Maine** | | **Maryland** | |
| 1957Q4 | | 1967 Q2 | 1984 Q3 | 1959 Q1 | 1984 Q1 | 1960 Q4 | 1984 Q1 | 1956 Q1 | 1978 Q3 | 1966 Q2 | 1988 Q1 | 1960 Q3 | 1988 Q1 |
| 1957Q4 | | 1967 Q2 | 1984 Q3 | 1959 Q1 | 1984 Q1 | 1960 Q4 | 1984 Q1 | 1956 Q1 | 1978 Q3 | 1966 Q2 | 1988 Q1 | 1960 Q3 | 1988 Q1 |
| **Massachusetts** | | **Michigan** | | **Minnesota** | | **Mississippi** | | **Missouri** | | **Montana** | | **Nebraska** | |
| 1959 Q1 | 1983 Q1 | 1957 Q4 | 1979 Q1 | 1968 Q1 | 1983 Q1 | 1956Q3 | | 1960 Q1 | 1981 Q3 | 1955 Q2 | 1981 Q3 | 1967 Q1 | 1981 Q2 |
| 1959 Q1 | 1983 Q1 | 1957 Q4 | 1979 Q1 | 1968 Q1 | 1983 Q1 | 1956Q3 | | 1960 Q1 | 1981 Q3 | 1955 Q2 | 1981 Q3 | 1967 Q1 | 1981 Q2 |
| **Nevada** | | **New Hampshire** | | **New Jersey** | | **New Mexico** | | **New York** | | **North Carolina** | | **North Dakota** | |
| 1962 Q1 | 1983 Q1 | 1959 Q1 | 1985 Q4 | 1959 Q1 | 1985 Q4 | 1963 Q2 | 1982 Q1 | 1963 Q2 | 1982 Q2 | 1958 Q3 | 1981 Q4 | 1962 Q4 | 1981 Q1 |
| 1962 Q1 | 1983 Q1 | 1959 Q1 | 1985 Q4 | 1959 Q1 | 1985 Q4 | 1963 Q2 | 1982 Q1 | 1963 Q2 | 1982 Q2 | 1958 Q3 | 1981 Q4 | 1962 Q4 | 1981 Q1 |
| **Ohio** | | **Oklahoma** | | **Oregon** | | **Pennsylvania** | | **Rhode Island** | | **South Carolina** | | **South Dakota** | |
| 1959 Q1 | 1984 Q4 | 1972 Q2 | 1981 Q3 | 1975Q4 | | 1958 Q3 | 1982 Q2 | 1958 Q1 | 1984 Q4 | 1959 Q1 | 1981 Q2 | 1960 Q4 | 1981 Q3 |
| 1959 Q1 | 1984 Q4 | 1972 Q2 | 1981 Q3 | 1975Q4 | | 1958 Q3 | 1982 Q2 | 1958 Q1 | 1984 Q4 | 1959 Q1 | 1981 Q2 | 1960 Q4 | 1981 Q3 |
| **Tennessee** | | **Texas** | | **Utah** | | **Vermont** | | **Virginia** | | **Washington** | | **West Virginia** | |
| 1957 Q3 | 1981 Q2 | 1958 Q1 | 1978 Q3 | 1959 Q2 | 1981 Q4 | 1959 Q1 | 1984 Q3 | 1963 Q3 | 1984 Q4 | 1959 Q1 | 1977 Q3 | 1958 Q2 | 1978 Q3 |
| 1957 Q3 | 1981 Q2 | 1958 Q1 | 1978 Q3 | 1959 Q2 | 1981 Q4 | 1959 Q1 | 1984 Q3 | 1963 Q3 | 1984 Q4 | 1959 Q1 | 1977 Q3 | 1958 Q2 | 1978 Q3 |

As an illustration we again observe the graph of Colorado in Fig. 4, which shows that the graph stabilises after the first change point (red dot) and it has a structural shift during the second change point (green dot).

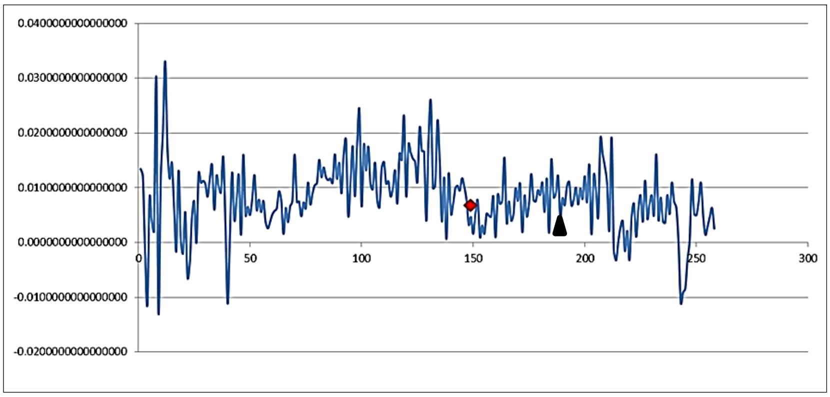


**Fig. 4. Graph of state Colorado with two break points**

The graphs of posterior distribution of break dates for each of the states as shown in Fig. 5 shows the location of change points for each of the 50 states of US over the years.

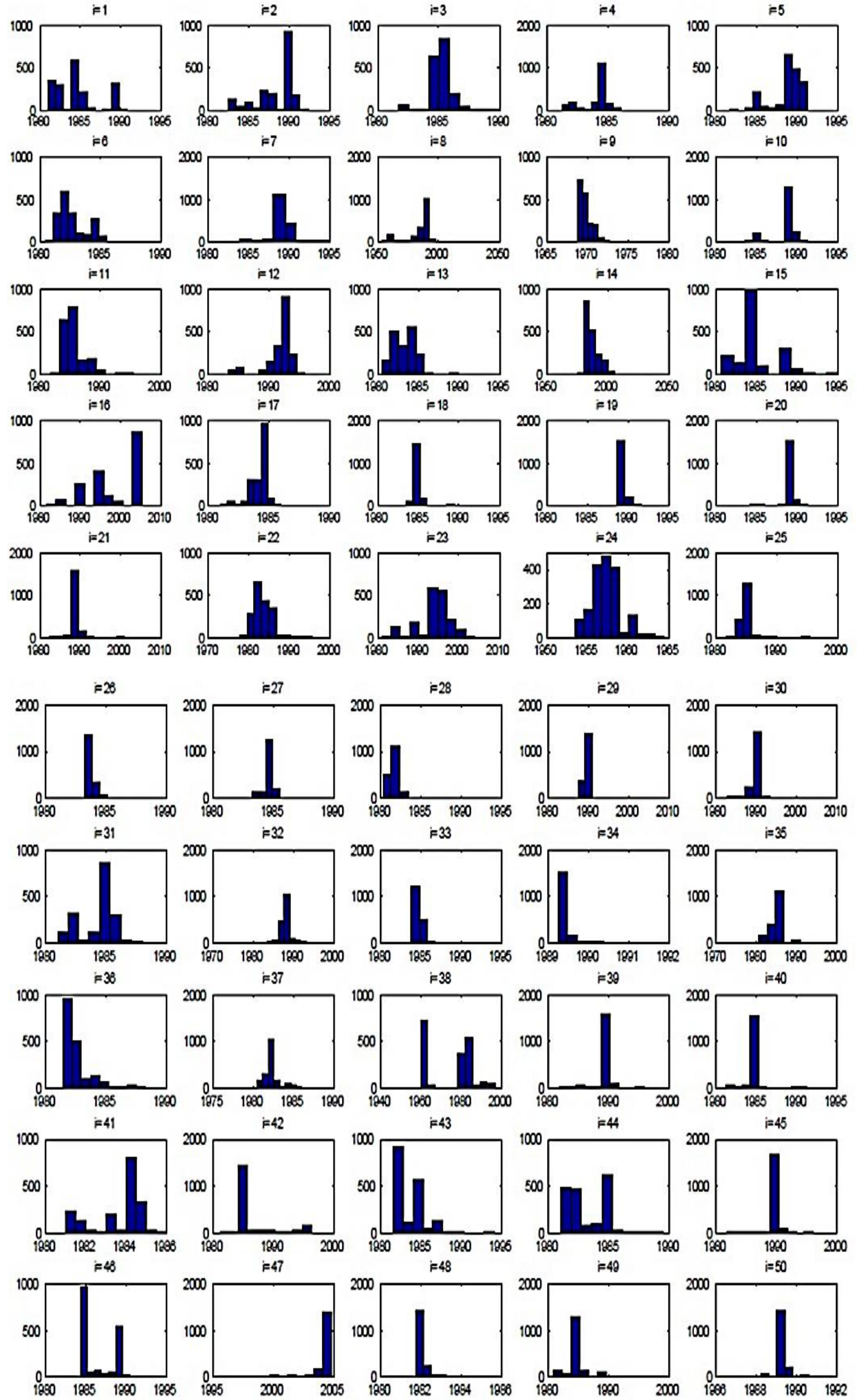


**Fig. 5. Histogram of the posterior distribution of the estimated break points for individual states**

## Conclusions

Most of the states have two change points with some exceptions. The graph of Colorado clearly depicted that structural breaks are more efficiently captured by our present model. The model takes into account the inter-dependency among the subjects i.e., interdependency among the data of each state which is a practical intrinsic assumption of every panel data. Bayesian model has low standard errors than the Classical model. States having single or no change points or change points much before 1984 or change points much after 1989 can be analyzed individually to inspect the fact whether changes were due to Great moderation or due to any other reason.

A two way random effects model can be implemented to eliminate the common factor present for all the subjects.

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