Exact Solutions of Stochastic Differential Equations: Gompertz and Generalized Logistic*

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Abstract. Exact analytic solutions of some stochastic differential equations are given along with characteristic futures of these models as the Mean and Variance. The procedure is based on the Ito calculus and a brief description is given. Classical stochastic models and also new models are provided along with a related bibliography. Stochastic models included are the Gompertz and the Generalized Logistic. Emphasis is given in the presentation of stochastic models with a sigmoid form for the mean value. These models are of particular interest when dealing with the innovation diffusion into a specific population, including the spread of epidemics, diffusion of information and new product adoption.
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1 Introduction

The Stochastic Differential Equations (SDE) play an important role in numerous physical phenomena. The numerical methods for solving these equations show low accuracy especially for the cases with high non-linear drift terms. It is therefore very important to search and present exact solutions for SDE. The resulting solutions are also important to check for the accuracy of existing numerical methods.

The first attempts for solving SDEs where based on proposing an integrating factor that could transform a SDE to a linear form that could be solved explicitly. A systematic method for reducing a non-linear SDE to a linear one was due to Kloeden and Platen, 1992 [7], and Kloeden et al 1999 [8], 2007 [9]. They proposed a suitable transformation function for the reduction of a particular SDE. This method is suitable for the cases presented here. The main theoretical issues are given in the following.

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2 Solution Methods of Stochastic Differential Equations

The method that will be presented and applied further down is based on the Ito norm [5], [6] and is used for the reduction of an autonomous nonlinear stochastic differential equation in the form of [7]:

$$dy(t) = a(y(t)) \cdot dt + b(y(t)) \cdot dw(t) \tag{1}$$

into a linear for $x(t)$ stochastic differential equation,

$$dx(t) = (a_1 \cdot x(t) + a_2) \cdot dt + (b_1 \cdot x(t) + b_2) \cdot dw(t) \tag{2}$$

Then the solution of this last equation is given by

$$x_t = \mathbf{\Phi}_t \left\{ x_0 + (a_2 - b_1 b_2) \int_0^t \mathbf{\Phi}_t^{-1} ds + b_2 \int_0^t \mathbf{\Phi}_t^{-1} dw_s \right\}$$

where

$$\mathbf{\Phi}_t = \exp\left\{ a_1 t - \frac{1}{2} b_1^2 t + b_1 w_t \right\}$$

By the use of a suitable transformation function $x(t) = U(y(t))$ the reduction method was initially presented by Gihman and Skorokhod (1972) [3] both for autonomous and for non-autonomous stochastic differential equations. In this case, only the reduction method for autonomous stochastic differential equations will be presented.

In applying this formula to Ito in the transformation function $U(y)$ the following results:

$$dU(y) = \frac{\vartheta U(y)}{\vartheta y} \cdot dy + \frac{1}{2} \cdot \frac{\vartheta^2 U(y)}{\vartheta y^2} \cdot (dy)^2$$

The above method will be used for the solution of some nonlinear stochastic diffusion equations which are presented in the following providing closed form solutions.

3 The Gompertzian Stochastic Model

The deterministic Gompertzian Model (Gompertz, 1825 [2]) has the form:

$$\frac{dx_t}{dt} = -b x_t \ln x_t$$

where b is a constant. This a growth model and the maximum growth rate is achieved when $x_{inf} = \exp(-1)$. This Gompertz function was proposed as a model to express the law of human mortality and can be used for population estimates. It is a sigmoid (S-shaped) model as regards the form x_t plotted against t. The Gompertz model is also applied in innovation diffusion modeling and in new product forecasting. In these cases the fluctuations are related to the magnitude x_t of the measurable characteristic part of the system and it is assumed that the noise term could be expressed by a multiplicative 1-dimensional white noise process. Thus the S.D.E. model resulting from the above deterministic one must have the form (see Skiadas et al [17]):

$$dx_t = -bx_t \ln x_t dt + cx_t dw_t$$

where x_t is the unknown stochastic process, b and c are constants and w_t is 1-dimensional Wiener process.

The solution of the last stochastic differential equation is obtained by applying the Ito formula to the transformation function $y_t = \ln x_t$ so that,

$$dy_t = d \ln x_t = x_t^{-1} dx_t - \frac{1}{2} x^{-2} (dx_t)^2$$

By substituting x_t from the above Gompertz stochastic differential equation and rearranging yields:

$$dy_t = d \ln x_t = (-by_t - \frac{1}{2} c^2) dt + c dw_t$$

The last equation is a stochastic linear differential equation and it is solved using the previous formulas to give the the following solution for x_t

$$x_t = \exp \left\{ \ln x_0 \exp(-bt) - \frac{c^2}{2b}(1 - \exp(-bt)) + c \exp(-bt) \int_0^t \exp(bs) dw_s \right\}$$

4 The Generalized Logistic Stochastic Model

A deterministic version of this model was developed by Richards (1959) [11] based on a previous simpler model proposed by Von-Bertalanffy for the description of the increase of weight as a function of the metabolism process of animals. Other deterministic forms of Generalized models can be found in [12], [13], [14].

4.1 The deterministic model

The deterministic Generalized Logistic model model is expressed by the differential equation

$$dx_t = bx_t \left(1 - \left(\frac{x_t}{F} \right)^m \right) dt$$

where b, m and F are parameters.

By dividing both sides of the last equation by F and placing $y_t = x_t / F$ results

$$dy_t = by_t\left(1 - (y_t)^m\right)dt$$

To solve this differential equation the method of change of variables is needed by using $z_t = y_t^{-m}$. Then the last differential equation reduces to the linear differential equation

$$dz_t = -bm(z_t - 1)dt \tag{3}$$

which is easily solved to give

$$\ln(z_t - 1) = \ln(z_0 - 1) = bmt \tag{4}$$

where $z_0 = z(t = 0) = y_0^{-m} = (x_0/F)^{-m}$

Finally by transforming to y_t and then to x_t the solution of the deterministic Generalized Logistic model results

$$x_t = F\left[1 + ((x_0/F)^{-m} - 1)\exp(-bmt)\right]^{-1/m} \tag{5}$$

This is a sigmoid form model with saturation level achieved at the upper limit F. The parameter b accounts for the speed of the product adoption process. The inflection point is achieved at $x_{inf} = (1/(m+1))^{1/m}$.

4.2 The stochastic model

The stochastic Generalized Logistic model with a multiplicative noise term is given by the stochastic differential equation

$$dx_t = bx_t\left(1 - \left(\frac{x_t}{F}\right)^m\right)dt + cx_t dw_t$$

As for the deterministic model above by dividing both sides of the last equation by F and placing $y_t = x_t/F$ results

$$dy_t = by_t\left(1 - (y_t)^m\right)dt + cy_t dw_t$$

For the solution of the last stochastic differential equation the reduction method will be used. The change of variables is achieved by using the same integration factor as for the deterministic case $z_t = y_t^{-m}$.

Then the Ito formula is applied to the transformation function z_t and by introducing the values for y_t and dy_t from the previous forms and rearranging the following form of the transformed stochastic differential equation results:

$$dz_t = \left(\left(\frac{c^2}{2}m(m+1) - bm\right)z_t + bm\right)dt - cmz_t dw_t$$

This is a linear autonomous stochastic differential equation. The solution arises after using the following general form for the solution of a linear stochastic differential equation of the type:

$$dr_t = (a_1 r_t + a_2)dt + (b_1 r_t)dw_t$$

where a_1, a_2 and b_1 are parameters. The solution is given by:

$$r_t = \Phi_t \left\{ r_0 + a_2 \int_0^t \Phi_t^{-1} ds \right\}$$

where $\Phi_t = \exp\left\{(a_1 - b_1^2/2)t + b_1 w_t\right\}$

Considering that in our case $a_1 = c^2 m(m+1)/2 - bm$, $a_2 = bm$ and $b_1 = -cm$; Φ_t is given by:

$$\Phi_t = \exp\{(-bm + c^2 m/2)t - cmw_t)\}$$

Then the resulting solution for z_t is

$$z_t = \Phi_t \left\{ z_0 + bm \int_0^t \Phi_s^{-1} ds \right\}$$

$$z_t = \Phi_t \left\{ z_0 + bm \int_0^t (\exp((-c^2 m/2 + bm)s + cmw_s))ds \right\}$$

Finally, the solution of the Generalized Logistic stochastic differential equation arises after the application of the reversal transformations $y_t = z_t^{-1/m}$ and $x_t = Fy_t$ and is in the form of:

$$y_t = \Phi_t^{-1/m} \left\{ y_0^{-m} + bm \int_0^t \Phi_s^{-1} ds \right\}^{-1/m}$$

$$x_t = F\Phi_t^{-1/m} \left\{ (x_0/F)^{-m} + bm \int_0^t \Phi_s^{-1} ds \right\}^{-1/m}$$

The resulting mean value for zero noise $c = 0$ is

$$x_t = F\left[1 + ((x_0/F)^{-m} - 1)\exp(-bmt)\right]^{-1/m} \tag{6}$$

That is precisely the solution of the deterministic case.

5 The Stochastic Logistic Model

The Logistic model, a model with very many applications in several fields, results as a special case of the Generalized Logistic model when the parameter $m = 1$. The stochastic version of this model is given by:

$$dx_t = bx_t \left(1 - \left(\frac{x_t}{F}\right)\right) dt$$

(See analytic solution and related applications in [4] and for a more general model in [15]) Then from the solution for the Generalized Logistic model the formula for the solution of the Logistic model results immediately by introducing $m = 1$.

$$x_t = F\Phi_t^{-1}\left\{(x_0/F)^{-1} + b\int_0^t \Phi_s^{-1} ds\right\}^{-1}$$

where

$$\Phi_t = \exp\left\{(-b + c^2/2)t - cw_t\right\}$$

$$x_t = F\Phi_t\left\{(x_0/F)^{-1} + b\int_0^t \exp\left\{(b - c^2/2)s + cw_s\right\} ds\right\}^{-1}$$

The resulting value for zero noise $c = 0$ is

$$x_t = F\left[1 + ((x_0/F)^{-1} - 1)\exp(-bt)\right]^{-1} \tag{7}$$

5.1 The mean value

To find the mean value of the Logistic stochastic model first we observe that the following relation holds for the expectation of a stochastic process $g(w)$

$$\mathbf{E}\{\exp(g(w))\} = \exp\frac{\mathbf{E}(g(w))^2}{2}$$

Thus the following two relations result

$$\mathbf{E}\{\exp(cw_t)\} = \exp\frac{\mathbf{E}(cw)^2}{2} = \exp\frac{\mathbf{E}\left(c^2 t\right)}{2} = \exp\frac{\left(c^2 t\right)}{2}$$

and

$$\mathbf{E}\left\{\exp\int_0^t (cw_s) ds\right\} = \exp\int_0^t \mathbf{E}\left\{(cw_s)^2/2\right\} ds = \exp\int_0^t (c^2 s/2) ds$$

We thus obtain

$$\mathbf{E}\{\Phi_t\} = \exp(-bt)$$

and

$$\mathbf{E}\left\{\exp\left(b\int_0^t \Phi_t^{-1} ds\right)\right\} = \mathbf{E}\left\{\exp\left(b\int_0^t ((b - c^2/2)s + cw_s) ds\right)\right\}$$
$$= \exp\left(b\int_0^t (bs) ds\right) = \exp(bt) - 1$$

Finally the mean value of the stochastic Logistic model is:

$$\mathbf{E}\{x_t\} = F/\left((x_0/F)^{-1}\exp(-bt) + (1-\exp(-bt)\right)$$

$$\mathbf{E}\{x_t\} = \frac{F}{1 + ((x_0/F)^{-1} - 1)\exp(-bt)}$$

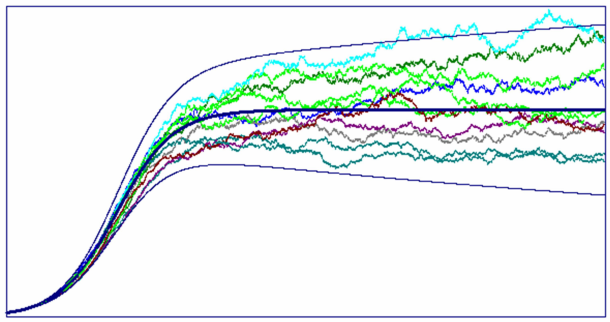

Fig. 1. The Stochastic Logistic Model

5.2 The Variance

To calculate the Variance of the stochastic Logistic model first we estimate the autocorrelation function that is

$$\mathbf{E}\{x_t x_t\} = F^2 \exp(2bt)\left\{(x_0/F)^{-1} + \exp(bt) - 1\right\}^{-2}$$

The Variance is calculated by the following form:
$\mathbf{Var}\{x_t\} = \mathbf{E}\{x_t x_t\} - (\mathbf{E}\{x_t\})^2$. Then

$$\mathbf{Var}\{x_t\} = F^2 \exp(2bt)\left\{(x_0/F)^{-1} + \exp(bt) - 1\right\}^{-2}(\exp(c^2 t) - 1)$$

We can now give an illustrative example of the Stochastic Logistic Model including several stochastic paths presented in Figure 1. The parameters selected are: $c = 0.0025$, $x_0 = 2$, $F = 100$, $b = 0.2$.

6 Conclusion

Exact solutions of several stochastic models are given along with the related analysis. The solution methods are based on the Ito theory for the solution of stochastic differential equations. Some sigmoid form models are given in the deterministic and the stochastic form and an illustrative example of the stochastic Logistic model is presented. The provided exact analytic forms could be very useful for testing the existing and new approximate methods for the solution of stochastic differential equations.

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